

# Example of how to work out a problem for the practical assignment Automated Reasoning 2IW15

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## Problem: eight queens problem

Find a way to put eight queens on a chess board such that no two queens share a column, row or diagonal.

## Solution:

We generalize this problem to putting  $n$  queens on an  $n \times n$  chess board, for any  $n \geq 1$ , with the same restriction that no two share a column, row or diagonal.

For doing so, we introduce  $n^2$  boolean variables  $p_{ij}$  for  $i, j = 1, \dots, n$ , where for every  $i, j = 1, \dots, n$  the value of  $p_{ij}$  will be true if and only if a queen will be put on position  $(i, j)$ , that is, in the  $i$ -th row and in the  $j$ -th column.

As we have to put exactly  $n$  queens, and no two are allowed to be on the same row, every row should contain at least one queen. For row  $i$  this is expressed by the formula

$$\bigvee_{j=1}^n p_{ij}.$$

In a similar way every column should contain at least one queen; for column  $j$  this is expressed by the formula

$$\bigvee_{i=1}^n p_{ij}.$$

Next we express that every row contains at most one queen, that is, for every two distinct positions  $j, k$  it is not allowed that both  $p_{ij}$  and  $p_{ik}$  are true. For row  $i$  this is expressed by the formula

$$\bigwedge_{j,k:1 \leq j < k \leq n} \neg p_{ij} \vee \neg p_{ik}.$$

Similarly, every column should contain at most one queen; for column  $j$  this is expressed by the formula

$$\bigwedge_{i,k:1 \leq i < k \leq n} \neg p_{ij} \vee \neg p_{kj}.$$

Finally, we consider the requirements on diagonals. Two positions  $(i, j)$  and  $(k, m)$  are on the same diagonal in the one direction if and only if  $i + k = j + m$ , and they are on the same diagonal in the other direction if and only if  $i - k = j - m$ . For every pair of such positions it is not allowed that they are both occupied by a queen, so we require

$$\neg p_{ij} \vee \neg p_{km}.$$

The total formula now consists of the conjunction of all these ingredients, that is,

$$\begin{aligned} & \bigwedge_{i=1}^n \left( \bigvee_{j=1}^n p_{ij} \right) \wedge \\ & \bigwedge_{j=1}^n \left( \bigvee_{i=1}^n p_{ij} \right) \wedge \\ & \bigwedge_{i=1}^n \left( \bigwedge_{j,k:1 \leq j < k \leq n} \neg p_{ij} \vee \neg p_{ik} \right) \wedge \\ & \bigwedge_{j=1}^n \left( \bigwedge_{i,k:1 \leq i < k \leq n} \neg p_{ij} \vee \neg p_{kj} \right) \wedge \\ & \bigwedge_{i,k:1 \leq i < k \leq n} \left( \bigwedge_{j,m:i+k=j+m \vee i-k=j-m} \neg p_{ij} \vee \neg p_{km} \right) \end{aligned}$$

This formula is easily expressed in SMT syntax, for instance, for  $n = 8$  one can generate

```
(benchmark test.smt
```

```
:extrapreds (
```

```
(p11) (p12) (p13) (p14) (p15) (p16) (p17) (p18)
```

```
(p21) (p22) (p23) (p24) (p25) (p26) (p27) (p28)
```

```
(p31) (p32) (p33) (p34) (p35) (p36) (p37) (p38)
```

```
(p41) (p42) (p43) (p44) (p45) (p46) (p47) (p48)
```

```
(p51) (p52) (p53) (p54) (p55) (p56) (p57) (p58)
```

```
(p61) (p62) (p63) (p64) (p65) (p66) (p67) (p68)
```

```
(p71) (p72) (p73) (p74) (p75) (p76) (p77) (p78)
```

```
(p81) (p82) (p83) (p84) (p85) (p86) (p87) (p88)
```

```
)
```

```
:formula
```

```
(and
```

```
(or (p11) (p12) (p13) (p14) (p15) (p16) (p17) (p18) )
```

```
(or (p21) (p22) (p23) (p24) (p25) (p26) (p27) (p28) )
```

```
.....
```

```
(or (not p11) (not p12))
```

```
(or (not p11) (not p13))
```

```
.....
```

```
))
```

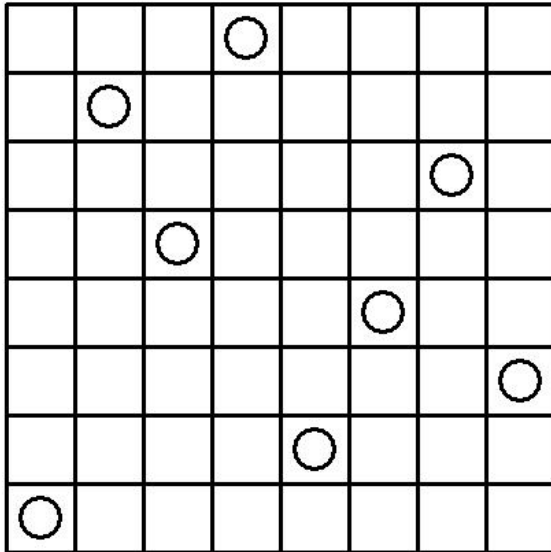
Applying `yices -e -smt test.smt` yields the following result within a fraction of a second:

```

sat
(= p11 false)
(= p12 false)
(= p13 false)
(= p14 true)
(= p15 false)
(= p16 false)
(= p17 false)
(= p18 false)
(= p21 false)
(= p22 true)
(= p23 false)
.....

```

The values that are true are  $p_{14}, p_{22}, p_{37}, p_{43}, p_{56}, p_{68}, p_{75}, p_{81}$ . Expressed in a picture this yields



We check that indeed there are eight queens for which no two are on the same row, column or diagonal.

**Remark:**

Our formula contains some redundancy: the requirement that every row contains exactly one queen implies that there are exactly  $n$  queens in total. By expressing that every column contains at least one queen, from this property one concludes that also every column contains at most one queen. We chose for this redundancy for keeping the symmetry in the solution, and also following the general strategy that introducing redundancy is often good for efficiency.

**Generalization:**

As we generalized our approach for  $n$  rather than 8, it is interesting to see what happens for other values of  $n$ . For  $n > 10$  we have to take care of the notation: if we keep the

notation then it is not clear whether `p111` represents  $p_{1,11}$  or  $p_{11,1}$ . This is solved by putting an extra symbol between the two numbers.

For  $n = 3$  the resulting formula is unsatisfiable, showing that there is no solution. For  $n = 4, 5, 6, \dots$  the formula is satisfiable, by which a solution of the problem is found. Efficiency is not a problem: for  $n = 60$  there are 3600 variables and the formula consists of over 350,000 clauses, but still `yices` succeeds in finding a solution within a few seconds.