Streams Coalgebraically

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CWI & RUN & VUA

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Streams are everywhere

- Psychology: Streams of consciousness
- Philosophy: You never step into the same stream twice
- Poetry: Dream streams and mean streams
- Politics: I had a stream ...
- Hydrology: M. Hauhs and H. Lange: Classification of runoff in headwater catchments: a physical problem? Geography Compass 2(1), 235-254, 2008.
- Computer science, mathematics, engineering: Data flow, Taylor series, signal processing, video streaming, etc. etc.

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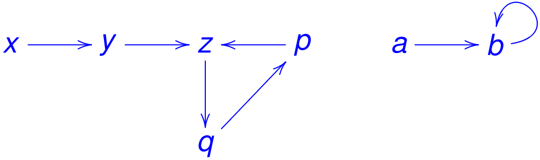
Overview

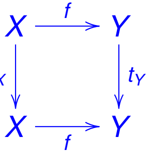
1. The method of coalgebra

- *basic*: dynamical systems
- systems with *output*: stream systems
- systems with *output* and *input*: automata
- systems with *output* and *input*: Mealy machines

Dynamical systems

- A dynamical system is: set X plus function $t : X \to X$
- Notation: $x \to y \quad \equiv \quad t(x) = y$. Example:



- *Homomorphisms*: 

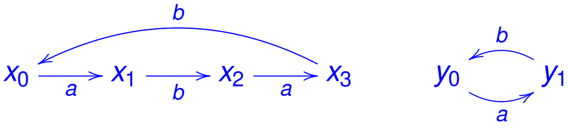
for reducing systems.

Adding output: stream systems

- A system with output in a set O:

$$< o, t >: X \to O \times X$$

- Notation: $x \xrightarrow{a} y \quad \equiv \quad o(x) = a \text{ and } t(x) = y$.



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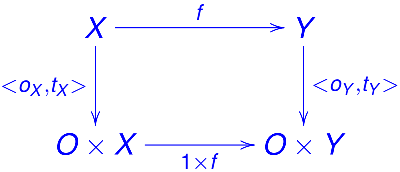


A final system with output: streams

- The set of streams: $O^\omega = \{\sigma \mid \sigma : \mathbb{N} \to O\}$ is *final*:

$$\begin{array}{ccc} X & \overset{\exists!\, f}{\dashrightarrow} & O^\omega \\ {\scriptstyle <o,t>}\downarrow & & \downarrow{\scriptstyle <head,\, tail>} \\ O \times X & \xrightarrow[1 \times f]{} & O \times O^\omega \end{array}$$

where streams transitions are given by:

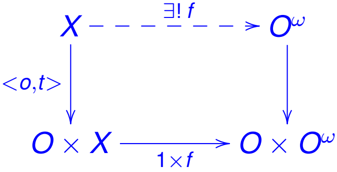
$$(\sigma(0), \sigma(1), \sigma(2), \ldots) \xrightarrow{\sigma(0)} (\sigma(1), \sigma(2), \sigma(3), \ldots)$$

and where the *final* homomorphism $f : X \to O^\omega$ is given by

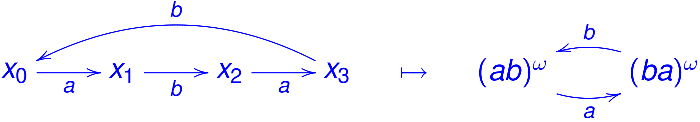
$$f(x) = (o(x), o(t(x)), o(t^2(x)), \ldots)$$

Streams: minimal systems with output

- *Finality* of the system of streams:

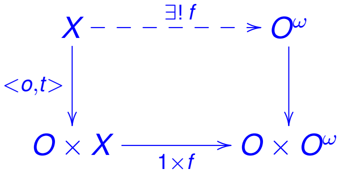


gives *semantics* in the form of *minimization*:



Defining streams by coinduction

- *Finality* gives also rise to definitions:



We say that $\sigma \in O^\omega$ is defined by *coinduction* if

$$\exists\,(X, < o, t >),\ \ \exists\, x \in X \ :\ \sigma = f(x)$$

- Too general, $(X, < o, t >)$ should be finite or *finitary*.
  To be continued in a minute.

Combining input and output: automata

- Language (of a state, of a regular expression):

$$
\begin{array}{ccc}
X & \overset{\exists!\, l}{- - - \to} & 2^{A^*} \\
\downarrow & & \downarrow \\
2 \times X^A & \longrightarrow & 2 \times (2^{A^*})^A
\end{array}
\qquad
\begin{array}{ccc}
RExp & \overset{\exists!\, l}{- - - \to} & 2^{A^*} \\
\downarrow{\scriptstyle \text{Brzozowski}} & & \downarrow \\
2 \times RExp^A & \longrightarrow & 2 \times (2^{A^*})^A
\end{array}
$$

given by *final* coalgebra homomorphism.

- Semantics $< l(x) > \subseteq 2^{A^*}$ is minimization of $< x > \subseteq X$.
- Language equivalence = coalgebraic bisimilarity
- Proofs of language equivalence: by coinduction.

Combining input and output: Mealy machines

- The set Γ of all *causal* functions

$$g : A^\omega \longrightarrow B^\omega$$

  is a *final Mealy machine*:

$$\begin{array}{ccc} X & \overset{\exists!\, h}{\dashrightarrow} & \Gamma \\ \downarrow & & \downarrow \\ (B \times X)^A & \longrightarrow & (B \times \Gamma)^A \end{array}$$

- Cf. second lecture.

Systems coalgebraically: summary

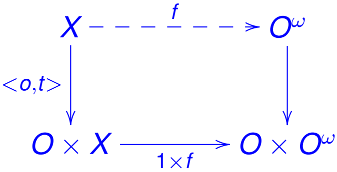
- Unified perspective on many types of systems:

$$
\begin{array}{cccc}
X & X & X & X \\
\downarrow & \downarrow & \downarrow & \downarrow \\
X & O \times X & 2 \times X^A & (B \times X)^A
\end{array}
$$

- *Homomorphisms* for comparing and reducing systems.
- *Final* systems give semantics by minimization and definitions by coinduction.
- Also *proofs* by coinduction = bisimulation proof method.

2. Defining streams by coinduction

- We say that $\sigma \in O^\omega$ is defined by *coinduction* if

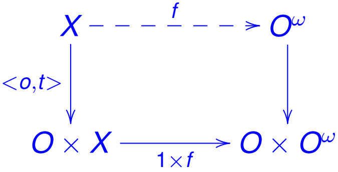


$$\exists\,(X, <o,t>),\ \ \exists\, x \in X\ :\ \sigma = f(x)$$

- If X is finite then $\sigma = f(x)$ is *eventually periodic*.

Coinductive stream definitions

Next we present two examples where X is infinite but *finitary*:



- Hamming numbers (here X will be an inductively defined term algebra)
- Rational streams (here X will be a finite dimensional vector space)

The Hamming numbers

All natural numbers, in increasing order, that have no other prime factors than 2 and 3:

$$\begin{aligned} \gamma &= (2^0 3^0,\, 2^1 3^0,\, 2^0 3^1,\, 2^2 3^0,\, 2^1 3^1,\, 2^3 3^0,\, 2^0 3^2,\, 2^2 3^1,\, \ldots) \\ &= (1, 2, 3, 4, 6, 8, 9, 12, \ldots) \end{aligned}$$

We define γ by a *stream differential equation*

$$\gamma' = (2 \times \gamma) \parallel (3 \times \gamma) \qquad \gamma(0) = 1$$

(where $\gamma' = \mathit{tail}(\gamma) = (\gamma(1), \gamma(2), \gamma(3), \ldots)$).

Our goal is to prove that this stream differential equation has *a (unique) solution*.

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The stream differential equation

$$\gamma' = (2 \times \gamma) \parallel (3 \times \gamma) \qquad \gamma(0) = 1$$

where the *ordered merge* $\parallel : \mathbb{N}^\omega \times \mathbb{N}^\omega \to \mathbb{N}^\omega$ is given by

$$(\sigma \parallel \tau)' = \begin{cases} \sigma' \parallel \tau & \text{if } \sigma(0) < \tau(0) \\ \sigma' \parallel \tau' & \text{if } \sigma(0) = \tau(0) \\ \sigma \parallel \tau' & \text{if } \sigma(0) > \tau(0) \end{cases}$$

$$(\sigma \parallel \tau)(0) = \begin{cases} \sigma(0) & \text{if } \sigma(0) < \tau(0) \\ \tau(0) & \text{if } \sigma(0) \geq \tau(0) \end{cases}$$

and where $2 \times \sigma$ (and similarly $3 \times \sigma$) satisfies

$$(2 \times \sigma)' = 2 \times (\sigma') \qquad (2 \times \sigma)(0) = 2 \cdot \sigma(0)$$

Solving the differential equation

$$\gamma' = (2 \times \gamma) \parallel (3 \times \gamma) \qquad \gamma(0) = 1$$

Assuming the solution exists, we compute the first few derivatives of γ:

$$\begin{aligned}
\gamma^{(1)} &= (2 \times \gamma) \parallel (3 \times \gamma) \\
\gamma^{(2)} &= (2 \times ((2 \times \gamma) \parallel (3 \times \gamma))) \parallel (3 \times \gamma) \\
\gamma^{(3)} &= (2 \times ((2 \times \gamma) \parallel (3 \times \gamma))) \parallel (3 \times ((2 \times \gamma) \parallel (3 \times \gamma)))
\end{aligned}$$

The idea: define *syntactic* terms for all possible such expressions:

$$\text{Term} \ni \ t ::= \text{c} \mid \underline{\sigma} \ (\sigma \in \mathbb{N}^{\omega}) \mid \text{2times}(t) \mid \text{3times}(t) \mid \text{merge}(t_1, t_2)$$

The syntactic stream system of terms

$\text{Term} \ni \; t ::= \; c \mid \underline{\sigma} \; (\sigma \in \mathbb{N}^\omega) \mid 2\text{times}(t) \mid 3\text{times}(t) \mid \text{merge}(t_1, t_2)$

Next we turn the set Term into a stream system

$$\text{Term} \xrightarrow{<o,n>} \mathbb{N} \times \text{Term}$$

by defining functions $o : \text{Term} \to \mathbb{N}$ and $n : \text{Term} \to \text{Term}$ by *induction* on the structure of terms, following the stream diff. eqn's. For instance,

$$n(\text{merge}(t_1,\, t_2)) = \begin{cases} \text{merge}(n(t_1),\, t_2) & \text{if } o(t_1) < o(t_2) \\ \text{merge}(n(t_1),\, n(t_2)) & \text{if } o(t_1) = o(t_2) \\ \text{merge}(t_1,\, n(t_2)) & \text{if } o(t_1) > o(t_2) \end{cases}$$

The solution

- By finality,

$$
\begin{array}{ccc}
\text{Term} & \overset{\exists\,!\,f}{- - - - - \to} & \mathbb{N}^\omega \\
\big\downarrow{\scriptstyle <o,n>} & & \big\downarrow \\
\mathbb{N} \times \text{Term} & \xrightarrow[1 \times f]{} & \mathbb{N} \times \mathbb{N}^\omega
\end{array}
$$

- Using f, we define

$$
\begin{aligned}
\gamma &= f(c) \\
\sigma \parallel \tau &= f(\text{merge}(\underline{\sigma}, \underline{\tau}))
\end{aligned}
$$

(and similarly for $2 \times \sigma$ and $3 \times \sigma$).

- Finally one shows that, indeed,

$$\gamma' = (2 \times \gamma) \parallel (3 \times \gamma) \qquad \gamma(0) = 1$$

Discussion

- An elementary and fairly general *syntactic* solution method.
- Many other examples. E.g., Thue-Morse sequence:

$$\sigma'' = \text{zip}(\sigma', \text{inv}(\sigma')) \quad \sigma(0) = 0, \;\; \sigma(1) = 1$$

  Also similar schemes for regular expressions, formal power series etc.

- *Format* of the right handside of the diff. eqn's is important.
- Interplay between algebra (terms, induction) and coalgebra (coinduction).
- Cf. λ-coiteration.

Rational streams

- Ingredients: vector space (here $\mathbb{R}^2$), and two linear maps:

$$O : \mathbb{R}^2 \to \mathbb{R}, \qquad T : \mathbb{R}^2 \to \mathbb{R}^2$$

- Finality as before, but everything now lives in *Vect* (category of vector spaces and linear maps):

$$\begin{array}{ccc} \mathbb{R}^2 & \overset{\exists!\, F}{\dashrightarrow} & \mathbb{R}^\omega \\ {\scriptstyle <O,T>}\downarrow & & \downarrow \\ \mathbb{R} \times \mathbb{R}^2 & \longrightarrow & \mathbb{R} \times \mathbb{R}^\omega \end{array}$$

- Note that $\mathbb{R}^2$ is *infinite* but O and T are *finite*.

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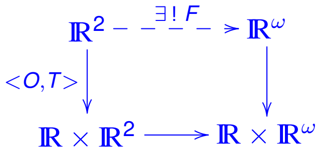

- Note that $\mathbb{R}^2$ is *infinite* but O and T are *finite*.

An example

- Let $O = \begin{pmatrix} 1 & 1 \end{pmatrix} : \mathbb{R}^2 \to \mathbb{R}$ and let

$$T = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} : \mathbb{R}^2 \to \mathbb{R}^2$$

- We can now compute F from O and T:

$$\begin{array}{ccc} \mathbb{R}^2 & \xrightarrow{\exists!\, F} & \mathbb{R}^\omega \\ {\scriptstyle <O,T>}\downarrow & & \downarrow \\ \mathbb{R} \times \mathbb{R}^2 & \longrightarrow & \mathbb{R} \times \mathbb{R}^\omega \end{array}$$

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using *stream calculus*: for all $(r_1, r_2) \in \mathbb{R}^2$,

$$\begin{aligned} F(r_1, r_2) &= O \circ (1 - (X \times T))^{-1} (r_1, r_2) \\ &= \begin{pmatrix} 1 & 1 \end{pmatrix} \circ \begin{pmatrix} \frac{1-2X}{(1-X)^2} & \frac{X}{(1-X)^2} \\ \frac{-X}{(1-X)^2} & \frac{1}{(1-X)^2} \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} \\ &= \frac{(r_1 + r_2) + X \times (-3r_1 + r_2)}{(1-X)^2} \end{aligned}$$

An example: conclusions

- We have computed F from O and T:

$$\begin{array}{ccc} \mathbb{R}^2 & \xrightarrow{\exists !\, F} & \mathbb{R}^\omega \\ {\scriptstyle <O,T>}\downarrow & & \downarrow \\ \mathbb{R} \times \mathbb{R}^2 & \longrightarrow & \mathbb{R} \times \mathbb{R}^\omega \end{array} \qquad F(r_1, r_2) = \frac{(r_1 + r_2) + X \times (-3r_1 + r_2)}{(1 - X)^2}$$

- More generally: a stream is *rational* if it can be obtained by coinduction from a finite dimensional vector space.
- Cf. Regular languages, formal power series.
- Cf. weighted automata.