Examination Automated Reasoning

with solutions

Code 2IMF25, January 18, 2016, 13.30 - 16.30

This examination consists of 5 problems each having the same weight. The final result for this course will be the average of the result for the practical assignment and the result for this examination, as long as both results are at least 5. Here for the practical assignment the average of both parts is taken.

Problem 1.

a. Compute the Tseitin transformation of

$$(p \leftrightarrow r) \rightarrow (r \land \neg q).$$

Solution:

Giving the name A to the whole formula, the name B to $p \leftrightarrow r$ and the name C to $r \land \neg q$, the Tseitin transformation consists of the following clauses:

A

$\neg A \lor \neg B \lor C$

$A \lor B$

$A \lor \neg C$

$B \lor p \lor r$

$\neg B \lor \neg p \lor r$

$B \lor \neg p \lor \neg r$

$\neg B \lor p \lor \neg r$

$\neg C \lor r$

$\neg C \lor \neg q$

$C \lor \neg r \lor q$

b. Explain how a verification problem given by a Hoare triple can be solved by SAT/SMT.

Solution:

A Hoare triple $\{P\}S\{Q\}$ states that if S is run from a state satisfying the precondition p, then after finishing the postcondition Q holds.

If S consists of doing m steps, then for every variable x in S and every $0 \leq i \leq m$ introduce x_i to denote the value of x after i steps. Write P_0 for the precondition P in which every variable is labeled by 0. Write Q_m for the postcondition Q in which every variable is labeled by m. Write $F(S)$ for the formula describing the semantics of S, expressed in x_i for x running over all variables and i running form 0 to m..

Then verification of the Hoare triple $\{P\}S\{Q\}$ is done by applying a SAT/SMT solver to

$$P_0 \ \wedge \ F(S) \ \wedge \ \neg Q_m.$$

If the SAT/SMT solver determines that the formula is unsatisfiable, then the verification is successful.

Problem 2.

Let x, y, z be real valued variables. Consider the CNF consisting of the following three clauses

$$\begin{array}{ll} (1) & y \leq x \vee z + 1 \leq x, \\ (2) & x + 1 \leq y \vee z \leq y, \\ (3) & x + 2 \leq z \vee y + 1 \leq z. \end{array}$$

Prove that this CNF is unsatisfiable by using the four rules UnitPropagate, Decide, Fail and Backtrack. Start by $(y \leq x)^d$. For the underlying steps based on the simplex algorithm the simplex algorithm does not need to be elaborated, for instance, you may immediately conclude that $y \leq x \wedge x + 1 \leq y$ yields a contradiction.

Solution:

Start by $(y \leq x)^d$.

Then by UnitPropagate on (2) we can conclude $z \leq y$.

As both literals in clause (3) contradict with the combination of $y \leq x$ and $z \leq y$, we may do a Backtrack, by which $z \leq y$ is removed and $(y \leq x)^d$ is replaced by its negation $\neg(y \leq x)$.

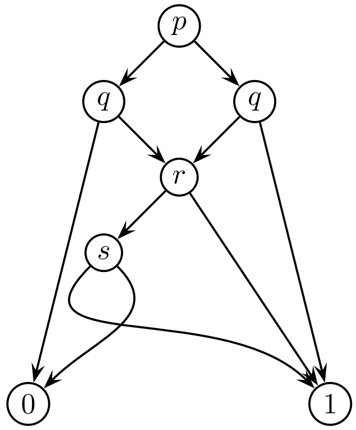
Then by UnitPropagate on (1) we conclude $z + 1 \leq x$.

As both literals in clause (3) contradict with the combination of $\neg(y \leq x)$ and $z + 1 \leq x$, and no decision literals occur, we do a Fail, proving that the formula is unsatisfiable.

Problem 3.

Compute the ROBDD of the boolean expression $2p+2q+r-s \leq 2$ with respect to the order $p < q < r < s$. Here false is identified with 0 and true with 1.

Solution:



Problem 4.

a. Compute $\sigma(x)$ for σ being the most general unifier of

$$f(f(g(z),z),f(x,z))$$

and

$$f(y,f(f(y,g(y)),g(w))).$$

Solution:

Unification yields no requirement for $\sigma(w)$, and

$$\sigma(z)=g(w),\ \sigma(y)=f(g(\sigma(z)),\sigma(z)),\ \sigma(x)=f(\sigma(y),g(\sigma(y))).$$

So

$$\sigma(x) \ = \ f(f(g(g(w)),g(w)),g(f(g(g(w)),g(w)))).$$

b. Transform

$$\exists y((\forall x P(x)) \rightarrow (\forall x(Q(x) \wedge P(y))))$$

to CNF by prenex normal form and skolemization.

Solution:

First we transforma to prenex normal form:

$$\begin{array}{ll} & \exists y((\forall x P(x)) \rightarrow (\forall x(Q(x) \wedge P(y)))) \\ = & \exists y((\forall x P(x)) \rightarrow (\forall z(Q(z) \wedge P(y)))) \\ = & \exists y((\neg \forall x P(x)) \vee (\forall z(Q(z) \wedge P(y)))) \\ = & \exists y((\exists x \neg P(x)) \vee (\forall z(Q(z) \wedge P(y)))) \\ = & \exists y \exists x \forall z(\neg P(x) \vee Q(z)) \wedge (\neg P(x) \vee P(y)) \end{array}$$

Applying Skolemizations means that y, x should be replaced by fresh constants c, d, yielding

$$(\neg P(d) \vee Q(z)) \wedge (\neg P(d) \vee P(c))$$

in which the variable z is implicitly universally quantified.

Problem 5.

The term rewriting system R is defined to consist of the rules

$$\begin{array}{rcl} f(g(x)) & \to & g(g(f(x))), \\ h(f(x)) & \to & g(x), \\ g(h(x)) & \to & h(h(x)). \end{array}$$

a. Prove that R is terminating by monotone interpretations.

Solution:

Take $[f](x) = 3x,\ [g](x) = x + 1,\ [h](x) = x$. Then

$$[f(g(x)), \alpha] = 3\alpha(x) + 3 > 3\alpha(x) + 2 = [g(g(f(x))), \alpha],$$

$$[h(f(x)), \alpha] = 3\alpha(x) > \alpha(x) + 1 = [g(x), \alpha],$$

$$[g(h(x)), \alpha] = \alpha(x) + 1 > \alpha(x) = [h(h(x)), \alpha],$$

for all natural numbers $\alpha(x) > 0$. Since $[f], [g], [h]$ are monotonic, this proves termination.

b. Give all non-trivial critical pairs of R.

Solution:

$f(g(h(x)))$ yields the critical pair

$$[f(h(h(x))), g(g(f(h(x))))].$$

$g(h(f(x)))$ yields the critical pair

$$[g(g(x)), h(h(f(x)))].$$

$h(f(g(x)))$ yields the critical pair

$$[h(g(g(f(x)))), g(g(x))].$$

c. Determine whether R is confluent.

Solution:

The first critical pair $[f(h(h(x))), g(g(f(h(x))))]$ consists of two distinct normal forms, so it does not converge. Hence R is not locally confluent, hence also not confluent.